NETS IN BANACH SPACES

Yuguang Xiao

M1 T.E.R.

Sorbonne Unversité

yuguang.xiao@etu.sorbonne-universite.fr

Encadrant: Alexandros Eskenazis

IMJ-PRG

Sorbonne Université

alexandros.eskenazis@imj-prg.fr

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Abstract

This paper revisits and systematizes existing results concerning Gromov's question on the bi-Lipschitz equivalence of arbitrary nets in Euclidean spaces. We reproduce known proofs, offer a unified presentation, and highlight that in infinite-dimensional Banach spaces, the problem admits an affirmative solution.

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1 Introduction

Definition 1 (separated net)**.** *If $a, b > 0$, an (a, b)-*net *or (a, b)-separated net in a normed space $(X, \|\cdot\|)$ is a subset $\mathcal{N} \subset X$ which is:*

- *a-*separated*, i.e. for every $u \neq v \in \mathcal{N}$, we have $\|u - v\| \geq a$;*
- *b-*dense*, i.e. for every $x \in X$ there exists $u \in \mathcal{N}$ such that $\|x - u\| \leq b$.*

Definition 2 (Bi-Lipschitz mapping)**.** *A map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is bi-Lipschitz if there is a constant K such that*

$$\frac{1}{K} < \frac{\|\phi(x) - \phi(x')\|}{\|x - x'\|} < K \tag{1}$$

for $x \neq x'$, where $\|\cdot\|$ is the Euclidean norm.

Definition 3 (Bi-Lipschitz equivalent subsets)**.** *Two subsets $\mathcal{N}_1, \mathcal{N}_2$ of a normed space $(X, \|\cdot\|)$ are bi-Lipschitz equivalent if there exists a bi-Lipschitz and bijective map from $\mathcal{N}_1$ to $\mathcal{N}_2$.*

In his investigation of geometric properties of groups, Gromov posed a fundamental question [Gro93]: **are any two nets in a given normed space X bi-Lipschitz equivalent?** While this is clearly true in $\mathbb{R}$, Burago–Kleiner [BK98] and McMullen [McM98] independently showed that it fails in $\mathbb{R}^n$ for $n \geq 2$. In contrast, Lindenstrauss, Matoušková, and Preiss [LKP00] found that in infinite-dimensional spaces, all nets are bi-Lipschitz equivalent, highlighting a key difference between finite and infinite dimensions. This project explores these results.

2 Preliminaries

First of all, we want to show that for two nets in $\mathbb{R}^n$, there exists a bijective map between them. We know that $\mathbb{Z}^n$ is a $(1, \frac{\sqrt{n}}{2})$-net in $\mathbb{R}^n$. It's enough to prove that there exist a bijective map from any net to $\mathbb{Z}^n$.

Theorem 1. *Let $\mathcal{N}$ be an (a, b)-net in $\mathbb{R}^n$. There exist a bijective map from $\mathcal{N}$ to $\mathbb{Z}^n$.*

Proof. Because $\mathcal{N}$ has the property that any two distinct points lie at least distance $a > 0$ apart, $\mathcal{N}$ is a *uniformly discrete* set. In a finite-dimensional Euclidean space $\mathbb{R}^n$, any such set is *locally finite*: each bounded region of $\mathbb{R}^n$ can only contain finitely many points from $\mathcal{N}$. Indeed, if infinitely many points of $\mathcal{N}$ lay in a bounded set, then by compactness they will have an accumulation point and this would contradict the uniform discreteness.

To see that $\mathcal{N}$ is at most countable, cover $\mathbb{R}^n$ by an ascending sequence of balls of radius $k = 1, 2, 3, \ldots$ around the origin,

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} B(0, k).$$

Since $\mathcal{N}$ has only finitely many points in each $B(0,k)$, we have

$$\mathcal{N} \;=\; \bigcup_{k=1}^{\infty} \big(\mathcal{N} \cap B(0,k)\big),$$

which is a countable union of finite sets, hence $\mathcal{N}$ is at most countable. In fact, $\mathcal{N}$ cannot be finite (because it must cover all of $\mathbb{R}^n$ in radius b), so $\mathcal{N}$ is *countably infinite*, as is $\mathbb{Z}^n$. □

3 The case of $\mathbb{R}^1$

It is not difficult to prove that Gromov's question holds in $\mathbb{R}^1$. Thanks to the preliminary section, we can know that there exist a bijection from (a,b)-net in $\mathbb{R}^1$ to $\mathbb{Z}^1$. We can write an (a,b)-net in $\mathbb{R}^1$ in the form of a sequence $(Z_n)_{n\in\mathbb{Z}}$.And then, we rearrange $(Z_n)_{n\in\mathbb{Z}}$ in ascending order to obtain a strictly increasing sequence $(Y_n)_{n\in\mathbb{Z}}$. We can show that the mini-spacing will be determined by the distance between two adjacent points.

We suppose that the increasing sequence $(Y_n)_{n\in\mathbb{Z}}$ is an (a,b)-net in $\mathbb{R}^1$.

Definition 4. *The distance between two adjacent points of $(Y_n)_{n\in\mathbb{Z}}$, note $d_a(Y_n)$:*

$$d_a(Y_n) = \{Y_{n+1} - Y_n : n \in \mathbb{N}\}$$

Theorem 2. *Let $(Y_n)_{n\in\mathbb{Z}}$ is a strictly increasing sequence, $(Y_n)_{n\in\mathbb{Z}}$ is a (a,b)-net in $\mathbb{R}^1$, if and only if* $\inf d_a(Y_n) \geq a$ *and* $\sup d_a(Y_n) \leq 2b$

Proof.

- We prove $\Longrightarrow$:
  1. Thanks to the propriety a-separated of (a,b)-net, we have $\inf d_a(Y_n) \geq a$.
  2. For the second proof, we use the proof by contradiction, we suppose that $\sup d_a(Y_n) > 2b$. There exists $k \in \mathbb{N}$, such that $Y_{k+1} - Y_k > 2b$. We pose a median point of the interval $[Y_k, Y_{k+1}]$, i.e: $x = \frac{Y_{k+1}+Y_k}{2} \in \mathbb{R}$, but $x - Y_k > b$. There is a contradiction.
- We prove $\Longleftarrow$:
  1. If $\inf d_a(Y_n) \geq a$, for all $m > k$, $(m,k) \in \mathbb{Z}\times\mathbb{Z}$, we have $Y_m - Y_k \geq (m-k)\cdot \inf d_a(Y_n) = (m-k)\cdot a \geq a$.
  2.

     Definition 5 (Neighbor of x in $(Y_n)_{n\in\mathbb{Z}}$). *The neighbor of x in the sequence $(Y_n)_{n\in\mathbb{Z}}$ is the set of points in $(Y_n)_{n\in\mathbb{Z}}$ that are at the minimum distance from x. We denote this set by N_x, i.e.,*

     $$N_x = \{Y_k : k \in \mathbb{Z}, \|Y_k - x\| \leq \|Y_m - x\| \text{ for all } m \neq k,\ m \in \mathbb{Z}\}.$$

$\#(N_x) = 2$ *if and only if* x *is a median point between* $[Y_{k+1}, Y_k]$ *for some* k. *Otherwise,* $\#(N_x) = 1$

For every $x \in \mathbb{R}^1$, there exists $N \in N_x$, we have $\|N - x\| \leq \frac{1}{2} \sup d_a(Y_n) = b$. □

Let $(Y_n)_{n\in\mathbb{Z}}$ and $(\tilde{Y_n})_{n\in\mathbb{Z}}$ be two (a, b)-nets in $\mathbb{R}^1$. We arrange Y_n along the horizontal axis and $\tilde{Y_n}$ along the vertical axis, forming a sequence of points $(Y_k, \tilde{Y_k})$ for $k \in \mathbb{Z}$. Looking at Figure 1, we then connect each adjacent pair of points with a simple line segment, thereby obtaining a piecewise linear map ϕ.

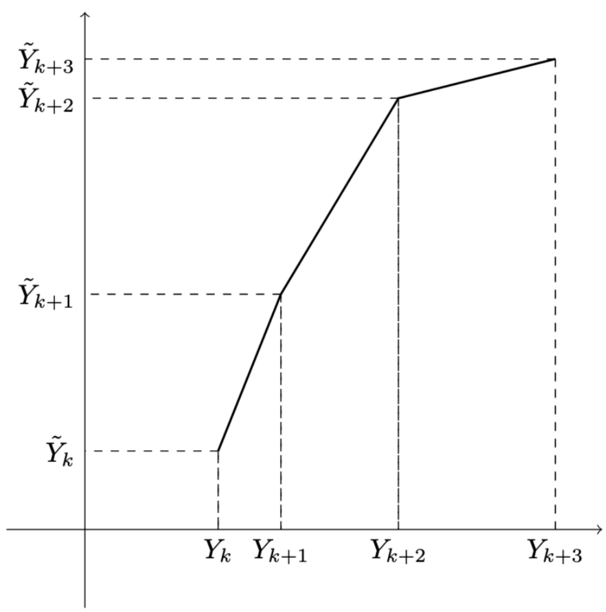

Figure 1: The simple map ϕ from $\{Y_n\}$ to $\{\tilde{Y_n}\}$

This map is necessarily bi-Lipschitz. By the preceding Theorem 2, we can easily find a constant $K = \frac{2b}{a}$, such that $\frac{1}{K} < \frac{\|\phi(x)-\phi(x')\|}{\|x-x'\|} < K$ for all $x \neq x' \in (Y_n)_{n\in\mathbb{Z}}$.

4 The case of $\mathbb{R}^n$, $n \geq 2$

In this chapter, we only consider the case $n \geq 2$.

4.1 Construction of an (a, b)-net

Theorem 3 (Rademacher Theorem)**.** *Let* U *be an open subset of* $\mathbb{R}^m$ *and let* $f : U \to \mathbb{R}^n$ *be a Lipschitz map. Then* f *is differentiable almost everywhere.*

Thanks to Rademacher Theorem, any bi-Lipschitz map have the Jacobian matrix. Let c be a strict positive constant, $I = [0, 1]$ and $\rho : I^2 \to [1, 1 + c]$ be a measurable function which is not the Jacobian of any bi-Lipschitz map $f : I^2 \to \mathbb{E}^2$ with

$$\text{Jac}(f) := \det(Df) = \rho \quad \text{a.e.}$$

Looking at Figure 2, we construct a sequence of disjoint square regions $\{S_k\}$ in $\mathbb{R}^2$, each aligned with the coordinate axes, with integer vertices. Each square S_k has side length l_k with $\frac{l_k}{l_{k-1}} = k$ growing to infinity as $k \to \infty$. Choose a sequence of positive integers $m_k \in]1, \infty[$ such that:

$$m_k \to \infty, \quad \frac{m_k}{l_k} \to 0 \quad as\ k \to \infty$$

(i.e, m_k grows slower than l_k).

Thus, l_k is much larger than m_k for large k. Let $I = [0, 1]$ and $\phi_k : I^2 \to S_k$ be the unique affine homeomorphism with scalar linear part because the side length of S_k is fixed. Then we define $\rho_k : S_k \to [\frac{1}{1+c}, 1]$ by $(\frac{1}{\rho}) \circ \phi_k^{-1}$. Subdivide S_k into m_k^2 sub-squares of side length $\frac{l_k}{m_k}$. Call this collection $\mathcal{T}_k = \{T_{ki}\}_{i=1}^{m_k^2}$.For each i in $\{1, \ldots, m_k^2\}$, subdivide T_{ki} into n_{ki}^2 sub-squares U_{kij} where n_{ki} is the integer part of $\sqrt{\int_{T_{ki}} \rho_k \, d\mathcal{L}}$ where $\mathcal{L}$ is Lebesgue measure.Let $\{O_n\}$ be the collection of squares of side length 1 that partition $(\cup S_k)^c$, and construct a discrete set $\mathcal{N} \subset \mathbb{R}^2$ by placing one point at the center of each O_n and at the center of each U_{kij}.

Lemma 1. *[BL98] $\mathcal{N}$ is an (a, b)-net in $\mathbb{R}^2$ where a and b need to be determined.*

Proof. • (a-separated)

We want to prove a-separated,(i.e, $\exists a \in \mathbb{R}_+, \inf_{u \neq v \in \mathcal{N}} \|u - v\| \geq a$). We partition $\mathcal{N}$ into two subsets, A and B. Here, A consists of all points contained in U_{kij}, while B consists of all points contained in $\{O_n\}$. Therefore, we only need to consider three cases:

1. the infimum of the distance between any two points within A:
   First of all, we consider the points in T_{ki} for some $i \in \{1, 2, ...m_k^2\}$. There exists $a_1 = \frac{l_k}{2m_k n_{ki}}$ such that
   $$\inf_{u \neq v \in T_{ki}} \|u - v\| \geq a_1 = \frac{l_k}{2m_k n_{ki}}$$
   Then we have $n_{ki} \leq \lfloor \sqrt{\int_{T_{ki}} 1 d\mathcal{L}} \rfloor \leq \sqrt{Area(T_{ki})}$, since ρ_k is bounded by 1. For each $i \in \{1, 2, ...m_k^2\}$, $\sqrt{Area(T_{ki})} = \frac{l_k}{m_k}$. So we find $a_1 \geq \frac{l_k}{2m_k \sqrt{Area(T_{ki})}} = \frac{1}{2}$.
   Secondly, we consider the points not in the same i of T_{ki}. We can easily find that $\frac{1}{2}$ is right for separation, because the point can not be at the boundary of T_{ki} for any i. The same thing in the different k.

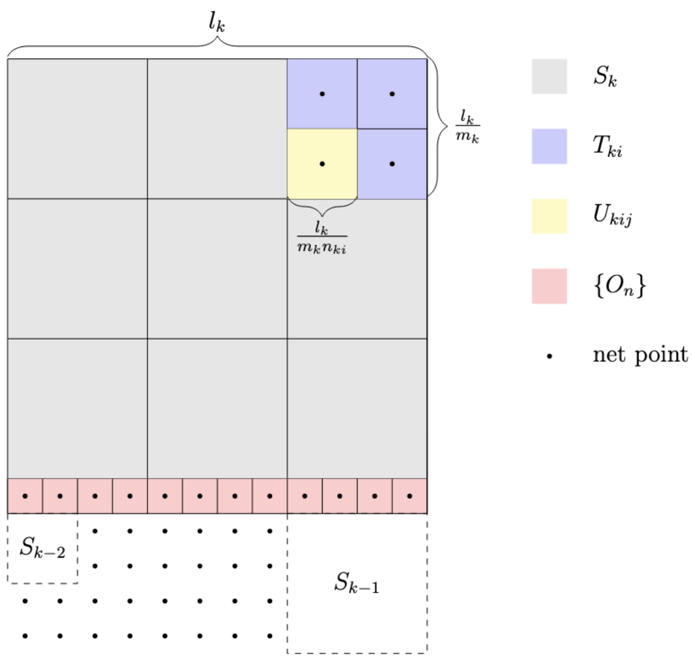

Figure 2: The sketch map of (a, b)-net $\mathcal{N}$ we construct for $k = 3$

2. the infimum of the distance between any two points within B:
   Thanks to a sub-set of $\mathbb{Z}^2$, $a_2 = 1$.
3. the infimum of the distance between any point in A and any point in B:
   We can easily find $a_3 = \frac{1}{2}$, because the point can not be at the boundary of S_k
   Conclusion: we finally find $a = min\{\frac{1}{2}, a_2, a_3\} = \frac{1}{2}$

- (b-dense)

  By the definition, we want to find $b \in \mathbb{R}_+$: for any $x \in \mathbb{R}^2$, there exists $u \in \mathcal{N}$, such that $\|x - u\| \leq b$. In this part, we just consider two cases:

  1. $x \in \cup S_k$:
     We want to find the below boundary of n_{ki}.

$$n_{ki} = \lfloor\sqrt{\int_{T_{ki}} \rho_k \, d\mathcal{L}}\rfloor \geq \lfloor\sqrt{\int_{T_{ki}} \frac{1}{1+c} \, d\mathcal{L}}\rfloor = \lfloor\sqrt{\frac{1}{1+c} \cdot Area(T_{ki})}\rfloor$$

$$> \sqrt{\frac{1}{1+c}} \cdot \sqrt{Area(T_{ki})} - 1 = \sqrt{\frac{1}{1+c}} \cdot \frac{l_k}{m_k} - 1$$

Then the side length of U_{kij} is $\frac{l_k}{m_k n_{ki}}$ which is bounded above by $\frac{l_k}{m_k} \cdot \frac{1}{\sqrt{\frac{1}{1+c} \cdot \frac{l_k}{m_k} - 1}} = \frac{1}{\sqrt{\frac{1}{1+c} - \frac{m_k}{l_k}}}$. By the definition, $\frac{m_k}{l_k}$ can reach its boundary above. So there exists $C_1 \in \mathbb{R}_+$ such that $\frac{l_k}{m_k n_{ki}} < C_1$. We get $b_2 = \frac{\sqrt{2}}{2} C_1$.

2. $x \in \cup O_n$:
   Thanks to B is a net in $\cup O_n$, like $\mathbb{Z}^2$ in $\mathbb{R}^2$, we can easily find $b_2 = \frac{\sqrt{2}}{2}$
   Conclusion: we finally find $b = max\{b_1, b_2\}$

□

4.2 Equivalent problem

Theorem 4. *Let $\rho : I^2 \to [1, 1+c]$ be a measurable function which is not the Jacobian of any biLipschitz map $f : I^2 \to \mathbb{R}^2$ with*

$$Jac(f) := \det(Df) = \rho \quad a.e.$$

Then there is a net in $\mathbb{R}^2$ which is not biLipschitz equivalent to $\mathbb{Z}^2$.

Proof. We now prove the theorem by contradiction. Suppose $g : \mathcal{N} \to \mathbb{Z}^2$ is an L-biLipschitz equivalence. Let $\mathcal{N}_k = \phi_k^{-1}(S_k \cap \mathcal{N}) \subset I^2$, and define $f_k : \mathcal{N}_k \to \mathbb{R}^2$ by

$$f_k(x) = \frac{1}{l_k} \left(g \circ \phi_k(x) - g \circ \phi_k(\star_k) \right)$$

where $\star_k$ is some basepoint in $\mathcal{N}_k$. Then f_k is an L-biLipschitz map from $\mathcal{N}_k$ to a subset of $\mathbb{E}^2$, and the f_k's are uniformly bounded. By the proof of the Arzelà-Ascoli theorem we may find a subsequence of the f_k's which "converges uniformly" to some biLipschitz map $f : I^2 \to \mathbb{R}^2$.

By the construction of $\mathcal{N}$, the counting measure on $\mathcal{N}_k$ (normalized by the factor $\frac{1}{l_k^2}$) converges weakly to $\frac{1}{\rho}$ times Lebesgue measure, while the (normalized) counting measure on $f_k(\mathcal{N}_k)$ converges weakly to Lebesgue measure. It follows that

$$f_* \left(\left(\frac{1}{\rho} \right) \mathcal{L} \right) = \mathcal{L}\big|_{f(I^2)}, \quad \text{i.e. Jac}(f) = \rho.$$

□

Theorem 5. *[McM98] For the following two statements, we have **(A)**, so **(B)**:*

(A) *Every measurable $f > 0$ on $\mathbb{R}^n$ with f and $1/f$ bounded can be realized as the Jacobian determinant of a bi-Lipschitz map.*

(B) *Every separated net $Y \subset \mathbb{R}^n$ is bi-Lipschitz to $\mathbb{Z}^n$.*

Proof.

Definition 6 (Voronoi cell of S)**.** *If S is a set of points in $\mathbb{R}^n$, called centers, we define the* Voronoi cell *of S as the set of points that are closer to a given point y than to any other point in S:*

$$V_y(S) = \{x \in \mathbb{R}^n \mid \|x - y\| < \|x - y'\| \text{ for all } y' \neq y \text{ in } S\}.$$

Definition 7 (diam(A))**.** *The* diameter *of a set $A \subset \mathbb{R}^n$ is defined as:*

$$diam(A) = \{\|x - y\| : x \neq y,\ (x, y) \in A \times A\}.$$

It represents a set of the distance between any two different points in A.

Definition 8 (vol(A))**.** *The* volume *of a set A in $\mathbb{R}^n$ refers to the measure of its size in n-dimensional space. In $\mathbb{R}^n$, volume represents the n-dimensional measure of A :*

$$vol(A) = \int_A d\mathcal{L}$$

Lemma 2. *Let $Y \subset \mathbb{R}^n$ be a (a, b)-net. For the Voronoi cell of Y :*

$$V_y(Y) = \{x \in \mathbb{R}^n \mid \|x - y\| < \|x - y'\| \text{ for all } y' \neq y \text{ in } Y\}.$$

We have:

- $M = \sup diam(V_y(Y)) < \infty$
- $m = \inf vol(V_y(Y)) > 0$

Proof.

- We first estimate the diameter of $V_y(Y)$:

$$\sup diam(V_y(Y)) = \sup_{x,x' \in V_y(Y)} \|x - x'\|.$$

Using the triangle inequality, we obtain:

$$\sup_{x,x' \in V_y(Y)} \|x - x'\| = \sup_{x,x' \in V_y(Y)} \|x - y + y - x'\|$$

$$\leq \sup_{x \in V_y(Y)} \|x - y\| + \sup_{x' \in V_y(Y)} \|y - x'\| \leq 2b < \infty.$$

- Next, we prove that $B(y, \frac{a}{2}) \subset V_y(Y)$. For any $x \in B(y, \frac{a}{2})$, we have $\|x - y\| < \frac{a}{2}$. Since Y is an (a, b)-net, for any $y' \neq y$, we have $\|y - y'\| \geq a$. Then, applying the triangle inequality:

$$2 \cdot \|x - y\| < a \leq \|y - y'\|.$$

Expanding using the triangle inequality:

$$\|y - y'\| = \|y - x + x - y'\| \leq \|x - y\| + \|x - y'\|.$$

Therefore, we obtain:

$$\|x - y\| < \|x - y'\|, \quad \forall y' \neq y.$$

This implies $x \in V_y(Y)$, meaning that $B(y, \frac{a}{2}) \subset V_y(Y)$. Consequently, we conclude:

$$0 < vol(B(y, \frac{a}{2})) \leq \inf vol(V_y(Y)).$$

□

Let

$$f(x) = \sum_{y \in Y} \frac{1}{vol(V_y(Y))} \mathbf{1}_{x \in V_y(Y)}$$

Then f is bounded by $\frac{1}{m}$ and $\frac{1}{f}$ bounded by M^n. Thanks to hypothesis **(A)**, it exists that a bi-Lipschitz map ϕ from $\mathbb{R}^n$ to $\mathbb{R}^n$, such that $det(jac(\phi)) = f$.

Lemma 3. *We define that $D_y = \phi(V_y(Y))$, we have:*

- *$vol(D_y) = 1$*
- *$diam(D_y)$ is bounded.*

Proof.

- $$vol(D_y) = \int_{D_y} d\mu = \int_{V_y(Y)} f d\mu = 1$$
- Thanks to the definition of bi-Lipschitz, we have:

$$\sup diam(D_y) = \sup_{u,v \in D_y} \|u - v\| = \sup_{u,v \in \phi(V_y(Y))} \|u - v\| \leq K \cdot \sup diam(V_y(Y)) < \infty$$

□

For $z \in \mathbb{Z}^n$, let E_z denote the unit cube centered at z. We have that $\sup diam(E_Z) = \sqrt{\sum_{k=1}^{n} 1^2} = \sqrt{n} < \infty$, so $diam(E_z)$ is bounded.

Definition 9 (The map R and the set $\mathcal{R}$)**.** *We define a R, from Y to $\mathbb{Z}^n$, such that:*

$$R(y) = \bigcup_{z \in \mathbb{Z}^n} \{z\} \cdot \mathbf{1}_{D_y \cap E_z \neq \phi}$$

We also definie a set $\mathcal{R} = \{(y, z) \in Y \times \mathbb{Z}^n : z \in R(y)\}$

Lemma 4. *For all $(y, z) \in \mathcal{R}$, we have $\|\phi(y) - z\|$ is bounded.*

Proof. For all $(y, z) \in \mathcal{R}$, by the previous definition, there exists a non-empty point $p \in \mathbb{R}^n$ such that $p \in D_y \cap E_z$. Then, we estimate:

$$\sup_{(y,z)\in\mathcal{R}} \|\phi(y) - z\| = \sup_{(y,z)\in\mathcal{R}} \|\phi(y) - p + p - z\|.$$

Using the triangle inequality, we obtain:

$$\sup_{(y,z)\in\mathcal{R}} \|\phi(y) - z\| \leq \sup_{y\in Y} \|\phi(y) - p\| + \sup_{(y,z)\in\mathcal{R}} \|p - z\|.$$

Since $p \in D_y \cap E_z$, it follows that:

$$\sup_{y\in Y} \|\phi(y) - p\| \leq \sup \operatorname{diam}(D_y), \qquad \sup_{(y,z)\in\mathcal{R}} \|p - z\| \leq \sup \operatorname{diam}(E_z).$$

Thus, we conclude:

$$\sup_{(y,z)\in\mathcal{R}} \|\phi(y) - z\| \leq \sup \operatorname{diam}(D_y) + \sup \operatorname{diam}(E_z) < \infty.$$

□

For any finite set $A \subset Y$, we have:

$$|A| = \operatorname{vol}\left(\bigcup_{y\in A} D_y\right),$$

since $\operatorname{vol}(D_y) = 1$ and the sets D_y tile $\mathbb{R}^n$.

Since each D_y is covered by $\bigcup_{z\in R(y)} E_z$, it follows that:

$$\operatorname{vol}\left(\bigcup_{y\in A} D_y\right) \leq \operatorname{vol}\left(\bigcup_{y\in A}\bigcup_{z\in R(y)} E_z\right) = \operatorname{vol}\left(\bigcup_{z\in R(A)} E_z\right).$$

Since $\operatorname{vol}(E_z) = 1$ and the sets E_z tile $\mathbb{R}^n$, we obtain:

$$\operatorname{vol}\left(\bigcup_{z\in R(A)} E_z\right) = |R(A)|.$$

Thus, we conclude:

$$|A| \leq |R(A)|.$$

Simmilary, $|R^{-1}(B)| \geq |B|$ for any finite set $B \subset \mathbb{Z}^n$

Theorem 6 (Hall's Marriage Theorem 1935). *[Mir71]*
A bipartite graph $G = (U, V; E)$ admits a perfect matching if and only if, for every subset X of U (or of V, respectively), the number of vertices in V (or in U, respectively) adjacent to at least one vertex in X is greater than or equal to the cardinality of X.

Here we consider that $U = Y$, $V = \mathbb{Z}^n$ and $E = \mathcal{R}$. For any finite set $A \subset Y$, we have $|R(A)| \geq |A|$. By the Hall's Marriage Theorem, there exists an injective map ψ_1 from Y to $\mathbb{Z}^n$. Simmilary, there exists an injective map ψ_2 from $\mathbb{Z}^n$ to Y.

Theorem 7 (Schröder–Bernstein theorem). *[Hal74]*
If there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$.

By the Schröder–Bernstein theorem, there exists a bijection $\psi : Y \to \mathbb{Z}^n$. Thanks to Lemma 4, we have that this map ψ is bi-Lipschitz. □

Theorem 8 (Equivalent theorem). *Let $\rho : I^2 \to [1, 1+c]$ be a measurable function which is the Jacobian of any biLipschitz map $f : I^2 \to \mathbb{R}^2$ with*

$$Jac(f) := \det(Df) = \rho \quad a.e.$$

If and only if for every net in $\mathbb{R}^2$ is biLipschitz equivalent to $\mathbb{Z}^2$.

Proof.
- We prove $\Longleftarrow$: The contraposition of Theorem 4.
- We prove $\Longrightarrow$: ρ and $\frac{1}{\rho}$ are bounded by $1 + c$ and apply Theorem 5.

□

4.3 The counterexample

In this part, we will find a counterexample in $\mathbb{R}^2$, which means:

Theorem 9 (Burago-Kleiner, McMullen). *There exists an $(a, b)-$net in $\mathbb{R}^2$ which is not bi-Lipschitz equivalent to $\mathbb{Z}^2$*

Proof. Before proving this theorem, we will see two lemmas first.

Lemma 5. *Let $\lambda > 1$ and $L > 1$. There exist constants $M \in \mathbb{N}$, $\eta > 0$, and $\mu > 1$ with the following properties:*

- $8L(\eta L + L/M) + \pi(\eta L + L/M)^2 < (\lambda - 1)/2$ *(Inequality 1),*
- $\mu^2 < 1 + \eta^2/5$ *(Inequality 2),*
- $(1 - \eta^2/5) + ((M + 1)^2 - 1)\mu < (M + 1)^2$ *(Inequality 3).*

Proof. Let $\eta = \frac{1}{M}$, where M is a sufficiently large natural number (to be determined). Define $\mu = 1 + \epsilon$, where $\epsilon > 0$ is a small parameter depending on M. We will choose ϵ to satisfy the inequalities.

Substitute $\eta = \frac{1}{M}$ into Inequality 1:

$$8L\left(\frac{L}{M} + \frac{L}{M}\right) + \pi\left(\frac{L}{M} + \frac{L}{M}\right)^2 = \frac{16L^2}{M} + \frac{4\pi L^2}{M^2}.$$

This simplifies to:

$$\frac{16L^2}{M} + \frac{4\pi L^2}{M^2} < \frac{\lambda - 1}{2}.$$

For sufficiently large M, both terms on the left become arbitrarily small. Thus, there exists $M_1 \in \mathbb{N}$ such that for all $M \geq M_1$, the inequality holds.

Substitute $\eta = \frac{1}{M}$ into Inequality 2:

$$\mu^2 < 1 + \frac{1}{5M^2}.$$

Let $\mu = 1 + \epsilon$. Then:

$$(1+\epsilon)^2 = 1 + 2\epsilon + \epsilon^2 < 1 + \frac{1}{5M^2}.$$

This requires:

$$2\epsilon + \epsilon^2 < \frac{1}{5M^2}.$$

If we choose $\epsilon = \frac{1}{10M^4}$, then:

$$2\epsilon = \frac{2}{10M^4} = \frac{1}{5M^4}, \quad \epsilon^2 = \frac{1}{100M^8}.$$

Thus:

$$2\epsilon + \epsilon^2 < \frac{1}{5M^4} + \frac{1}{100M^8} < \frac{1}{5M^2} \quad \text{for } M \geq 1.$$

Hence, $\mu = 1 + \frac{1}{10M^4}$ satisfies Inequality 2.

Substitute $\eta = \frac{1}{M}$ and $\mu = 1 + \frac{1}{10M^4}$ into Inequality 3:

$$\left(1 - \frac{1}{5M^2}\right) + \left[(M+1)^2 - 1\right]\left(1 + \frac{1}{10M^4}\right) < (M+1)^2.$$

Expand the left-hand side (LHS):

$$1 - \frac{1}{5M^2} + (M^2 + 2M)\left(1 + \frac{1}{10M^4}\right).$$

$$= 1 - \frac{1}{5M^2} + (M^2 + 2M) + \frac{M^2 + 2M}{10M^4}.$$

$$= (M+1)^2 - \frac{1}{5M^2} + \frac{1}{10M^2} + \frac{2}{10M^3}.$$

$$= (M+1)^2 - \frac{1}{10M^2} + \frac{1}{5M^3}.$$

Since $-\frac{1}{10M^2} + \frac{1}{5M^3} < 0$ for $M \geq 2$, we have:

$$\text{LHS} < (M+1)^2.$$

Thus, Inequality 3 holds for $M \geq 2$.
Finally, choose $M = \max\{M_1, 2\}$, $\eta = \frac{1}{M}$, and $\mu = 1 + \frac{1}{10M^4}$. By construction:

- Inequality 1 holds for $M \geq M_1$,
- Inequality 2 holds for $\mu = 1 + \frac{1}{10M^4}$,
- Inequality 3 holds for $M \geq 2$.

Therefore, such M, η, and μ exist.

□

These constants will be used to prove the statement below.

Lemma 6. *[BL98] Let $\lambda > 1$ and $L > 1$ be arbitrary constants. Then there exists a constant $\mu > 1$ such that the following holds.*

Let $x, y \in \mathbb{R}^2$ be two distinct points, and let U be an open neighborhood of the closed segment $[x, y] \subset \mathbb{R}^2$.

Then, one can construct:

- *a measurable function $f : U \to \{1, \lambda\}$,*
- *a finite collection of pairwise disjoint closed segments $I_j = [x_j, y_j] \subset U$, and*
- *a small constant $\varepsilon > 0$,*

with the following property:

Suppose that $\varphi : U \to \mathbb{R}^2$ is a bi-Lipschitz homeomorphism (i.e., a Lipschitz bijection whose inverse is also Lipschitz), such that:

- *the Lipschitz constants of both φ and φ^{-1} are bounded by L, and*
- *the Jacobian determinant $J(\varphi)$ differs from f only on a set of small measure:*

$$\mathcal{L}\{u \in U : J(\varphi)(u) \neq f(u)\} < \varepsilon,$$

where $\mathcal{L}$ denotes the Lebesgue measure.

Then there exists an index j such that:

$$\frac{\|\varphi(x_j) - \varphi(y_j)\|}{\|x_j - y_j\|} \geq \mu \cdot \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|}.$$

Proof. First, thanks to Lemma 5, choose a positive integer M and $\eta > 0$ such that

$$8L(\eta L + L/M) + \pi(\eta L + L/M)^2 < \frac{\lambda - 1}{2}.$$

Next, pick $\mu > 1$ so that $\mu^2 < 1 + \eta^2/5$ and

$$(1 - \eta^2/5) + \big((M+1)^2 - 1\big)\mu < (M+1)^2.$$

We use a suitable scaling and choice of coordinates so that U is a neighborhood of the line segment connecting $x = (0,0)$ and $y = (1,0)$. For a sufficiently large integer N (later we will see further constraints on N), the set U contains the rectangle

$$R = \{(s,t) : 0 \leq s \leq 1,\ 0 \leq t \leq 1/N\}.$$

We partition R into N adjacent squares, looking at Figure 3:

$$S_i = \{(s,t) : (i-1)/N \leq s \leq i/N,\ 0 \leq t \leq 1/N\}, \quad i = 1, 2, \ldots, N.$$

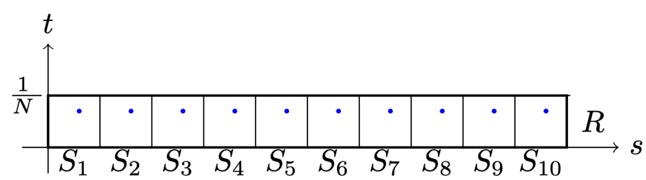

Figure 3: For $N = 10$, the sketch map of R, S_i and $z_{k,l}$ for some k and l (blue points)

Define a function f on R such that $f = 1$ on S_i when i is even, and $f = \lambda$ on S_i when i is odd.

Next, introduce a grid of points at spacing $(MN)^{-1}$. For each $1 \le i \le N$, $0 \le k < M$, and $0 \le l \le M$, define

$$z_{k,l}(i) = \left(\frac{i-1}{N} + \frac{k}{MN},\ \frac{l}{MN}\right).$$

Let $\varphi : U \to \mathbb{R}^2$ be a homeomorphism whose Lipschitz constant and that of its inverse are both bounded by L. Without loss of generality, assume:

$$\varphi(x) = (0,0) \quad \text{and} \quad \varphi(y) = (a,0)$$

for some $a > 0$. For each $1 \le i \le N-1$, set

$$w_{k,l}(i) = \varphi\big(z_{k,l}(i+1)\big) - \varphi\big(z_{k,l}(i)\big).$$

Our goal is to show that if ε is sufficiently small and

$$\mathcal{L}\{x \in U : J(\varphi)(x) \ne f(x)\} < \varepsilon,$$

then there must be some choice of (k,l,i) with

$$\|w_{k,l}(i)\| \ \ge\ \frac{\mu a}{N}.$$

This proves the lemma because one can then separate the image of the intervals

$$J_{k,l,i} = [\, z_{k,l}(i),\ z_{k,l}(i+1)\,]$$

(which are all parallel to the x-axis) by a small displacement, using the Lipschitz nature of φ.

Assume, for the sake of contradiction, that

$$\|w_{k,l}(i)\| \ \le\ \frac{\mu a}{N} \quad \text{for all } k,l,i.$$

We will prove that there is some index i for which all k,l satisfy

$$\|\, w_{k,l}(i) - w\,\| \ \le\ \frac{\eta a}{N},$$

where $w = (a/N,\ 0)$. In other words, $w_{k,l}(i)$ is close to the pure horizontal displacement w.

Let $\rho_{k,l}(i)$ be the length of the projection of $w_{k,l}(i)$ onto the x-axis. Since $\|w_{k,l}(i)\| \le \mu a/N$ and $\mu^2 < 1 + \eta^2/5$, we only need to show that some i makes

$$\rho_{k,l}(i) \ \ge\ \Big(1 - \frac{\eta^2}{5}\Big)\frac{a}{N} \quad \text{for all } k,l.$$

Suppose no such i exists. There are indices $0 \le r < M$ and $0 \le s \le M$, by the pigeonhole principle:

- $i = 1 \to (k_1, l_1) \in \{0, 1, ..., M\}^2$
- $i = 2 \to (k_2, l_2)$
  ......
- $i = N \to (k_N, l_N)$

There exists $\lfloor \frac{N}{(M+1)^2} \rfloor$ many i's such that (k_j, l_j) is the same point to (r, s). Then there is a subset $I \subset \{0, 1, \ldots, N-1\}$ of size

$$|I| \;\geq\; \frac{N}{(M+1)^2}$$

such that for every $i \in I$,

$$\rho_{r,s}(i) \;<\; \Big(1 - \frac{\eta^2}{5}\Big)\frac{a}{N}.$$

Looking at Figure 4, consider the polygonal path formed by connecting $(0,0)$, $(0, \frac{s}{MN})$, $z_{r,s}(1)$, $z_{r,s}(2)$, ..., $z_{r,s}(N)$, $(1, \frac{s}{MN})$, and finally $(1, 0)$. Its image under φ goes from $(0,0)$ to $(a, 0)$, so the projection of this image on the x-axis has length at least a.

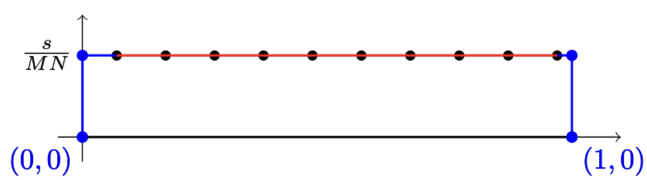

Figure 4: In the polygonal path we constructed, the black points represent $Z_{r,s}(i)$ for $i \in \{1, 2, \ldots, N\}$, while the blue points correspond to $(0,0)$, $\big(0, \frac{s}{MN}\big)$, $\big(1, \frac{s}{MN}\big)$, and $(1, 0)$.

On I, the horizontal projections of $w_{r,s}(i)$ are less than $(1-\eta^2/5)(a/N)$; on the complement of I, they are at most $\mu a/N$. In addition, by the L-Lipschitz property, the two vertical segments and the first and last horizontal segments (i.e., the two blue lines parallel to the x-axis) each have images of length at most L/N. The remaining two blue lines also have images of length at most L/N each. Combining these estimates, the total projected length on the x-axis becomes:

$$a \;\leq\; \frac{N}{(M+1)^2}\Big(1 - \frac{\eta^2}{5}\Big)\frac{a}{N} \;+\; \Big(N - \frac{N}{(M+1)^2}\Big)\frac{\mu a}{N} \;+\; \frac{4L}{N}.$$

For large N, this contradicts our choice of μ (because of the condition $(1-\eta^2/5)+((M+1)^2-1)\mu < (M+1)^2$). Hence our assumption fails, and we can pick i with the desired property

$$\| \, w_{k,l}(i) - w \, \| \;\leq\; \frac{\eta a}{N} \quad \text{for all } k, l.$$

Now compare the areas of $\varphi(S_i)$ and $\varphi(S_{i+1})$. Note that translating $\varphi(S_i)$ by w does not change its area. Also,

$$\left\|(\varphi(z_{k,l}(i)) + w) - \varphi(z_{k,l}(i+1))\right\| \leq \frac{\eta a}{N}, \tag{2}$$

Definition 10 (Hausdorff distance). *Let (M, d) be a metric space. Looking at Figure 5, for each pair of non–empty subsets $X \subset M$ and $Y \subset M$, the Hausdorff distance between X and Y is defined as*

$$d_H(X,Y) := \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(X,y)\right\},$$

where

$$d(a,B) := \inf_{b\in B} d(a,b)$$

quantifies the distance from a point $a \in X$ to the subset $B \subseteq X$.

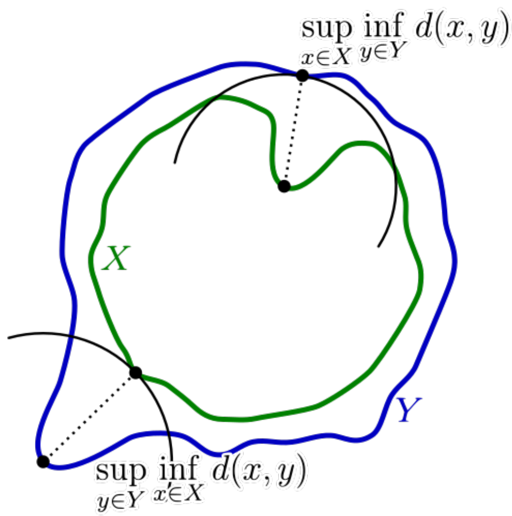

Figure 5: Components of the calculation of the Hausdorff distance between the green curve X and the blue curve Y.

Thanks to (2), each pair of corresponding points are within a distance of at most $\frac{\eta a}{N}$. Looking at Figure 6, every point lies within $\frac{L}{MN}$ of the boundary (we can move the second point on the boundary). Thanks to triangle inequality, the sets $\varphi(S_i) + w$ and $\varphi(S_{i+1})$ have boundaries at most $\eta a/N + L/(MN)$ apart in the Hausdorff sense.

Definition 11 (δ-neighborhood). *Let γ be a curve in $\mathbb{R}^2$, parameterized by arc length and of*

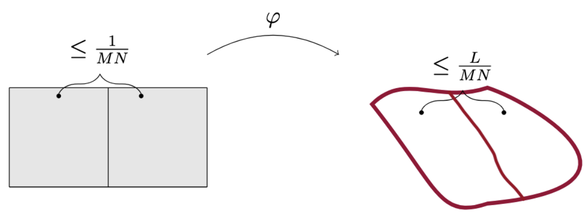

Figure 6: Two points in S_i and S_{i+1} are transferred via φ

total length l. The δ-neighborhood *of γ is defined as*

$$N_\delta(\gamma) := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \gamma) < \delta\},$$

Claim 1. *Let γ be a curve in $\mathbb{R}^2$ of length l. Then for any $\delta > 0$, the area of its δ-neighborhood satisfies the upper bound:*

$$\mathcal{L}(N_\delta(\gamma)) \leq 2\delta l + \pi\delta^2.$$

Since each boundary has length at most $4L/N$, the standard upper bound on the measure of a δ-neighborhood shows

$$|\mathcal{L}(\varphi(S_i)) - \mathcal{L}(\varphi(S_{i+1}))| \;=\; |\mathcal{L}(\varphi(S_i)+w) - \mathcal{L}(\varphi(S_{i+1}))| \;\leq\; 8\Big(\frac{L}{N}\Big)\Big(\frac{\eta a}{N} + \frac{L}{MN}\Big) + \pi\Big(\frac{\eta a}{N} + \frac{L}{MN}\Big)^2.$$

Since $a \leq L$, our initial choice of M and η implies

$$8\Big(\frac{L}{N}\Big)\Big(\eta\frac{L}{N} + \frac{L}{MN}\Big) + \pi\Big(\eta\frac{L}{N} + \frac{L}{MN}\Big)^2 \;<\; \frac{\lambda - 1}{2N^2}.$$

Therefore

$$\big|\,\mathcal{L}(\varphi(S_i)) - \mathcal{L}(\varphi(S_{i+1}))\big| < \frac{\lambda - 1}{2N^2}.$$

By the definition of Area:

$$\mathcal{L}(\varphi(S_i)) = \int_{S_i} Jac(\varphi) d\mathcal{L} = \int_{S_i \cap \{Jac(\varphi) = f\}} \mathbf{1}_{\{i\ even\}} + \lambda_{\{i\ odd\}} d\mathcal{L} + \int_{S_i \cap \{Jac(\varphi) \neq f\}} Jac(\varphi) d\mathcal{L}$$

Now suppose i is even. By construction, $f = 1$ on S_i and $f = \lambda$ on S_{i+1}, except on sets of total measure at most ε. Because $J(\varphi) = f$ almost everywhere outside a set of measure ε, we get

$$\mathcal{L}(\varphi(S_i)) \;\leq\; \frac{1}{N^2} \;+\; \varepsilon L^2, \quad \mathcal{L}(\varphi(S_{i+1})) \;\geq\; \frac{\lambda}{N^2} \;-\; \varepsilon L^2.$$

Hence,

$$\mathcal{L}(\varphi(S_{i+1})) - \mathcal{L}(\varphi(S_i)) \;\geq\; \frac{\lambda-1}{N^2} \;-\; 2\varepsilon L^2.$$

On the other hand, we have just shown

$$\mathcal{L}(\varphi(S_{i+1})) - \mathcal{L}(\varphi(S_i)) \;<\; \frac{\lambda-1}{2N^2},$$

leading to

$$\frac{\lambda-1}{2N^2} > \frac{\lambda-1}{N^2} - 2\,\varepsilon\,L^2 \implies (\lambda-1)\,\bigl(\tfrac{1}{2}-1\bigr)\,\frac{1}{N^2} \;<\; -2\,\varepsilon\,L^2.$$

This is impossible if

$$\varepsilon \;<\; \frac{\lambda-1}{4\,N^2\,L^2}.$$

Hence our assumption must be false, completing the proof. □

We begin by fixing a real number $\lambda > 1$, and consider a countable family of pairwise disjoint open sets $\{U_L : L = 1, 2, \ldots\}$ contained in some domain $B \subset \mathbb{R}^2$. Our goal is to assign, for each L, a function

$$f_L : U_L \;\to\; \{1, \lambda\}$$

with the property that no bi-Lipschitz map

$$\varphi \;:\; U_L \;\longrightarrow\; \mathbb{R}^2$$

can have Lipschitz constant (and inverse Lipschitz constant) bounded by L while simultaneously satisfying $J(\varphi) = f_L$ throughout U_L. Here $J(\varphi)$ denotes the Jacobian determinant of φ. Eventually, we define a global function f on B by letting $f = f_L$ on each U_L and setting $f = 1$ outside $\bigcup_L U_L$.

Choose a parameter $\mu = \mu(\lambda, L)$ according to a Lemma 6, ensuring certain stretching properties. Let k be an integer such that $\mu^k/L > L$. By repeated use of the lemma, we inductively construct:

- Finite collections of disjoint line segments $\{I_{n,j}\}$ in U_L, for $1 \leq n \leq k$,
- Positive numbers $\varepsilon_n > 0$,
- Functions g_n defined on U_L,
- Disjoint open sets $U_{n,j} \supset I_{n,j}$,

all built step by step as follows.

Take two interior points $x, y \in U_L$. By applying the Lemma 6 once, we obtain a function g_1 on U_L, a finite family of disjoint segments $\{I_{1,j}\}$, and a small $\varepsilon_1 > 0$. We then choose disjoint open sets $U_{1,j}$ (each containing $I_{1,j}$) so that the total area of $\bigcup_j U_{1,j}$ is under $\varepsilon_1/2$.This ensures that the measure of the set where Jac is not equal to g_1 is less than ε_1. This concludes the first iteration.

Next, we restrict attention to each open set $U_{1,j}$ individually. On $U_L \setminus \bigcup_j U_{1,j}$ we keep the function g_1, and we refine it inside each $U_{1,j}$ as follows:

- For a given $I_{1,j}$, we again apply the lemma with $I_{1,j}$ as endpoints and $U_{1,j}$ as domain to get a new function $g_{1,j}$, a smaller $\varepsilon_{1,j}$, and a disjoint family of segments $\{I_{1,j,k}\}$.
- We define g_2 to be $g_{1,j}$ within $U_{1,j}$; set $\varepsilon_2 = \min_j \varepsilon_{1,j}$.
- Around each $I_{1,j,k}$, choose a slightly larger open set $U_{1,j,k} \subset U_{1,j}$ of small total area (so that $\bigcup_{j,k} U_{1,j,k}$ has area less than $\varepsilon_2/2$).

We then renumber all $I_{1,j,k}$ as $I_{2,j}$ and similarly $U_{1,j,k}$ as $U_{2,j}$. This completes the second step.

We repeat this procedure k times, each time refining the function inside the newly-created open sets. After k such iterations, the final function g_k is declared to be f_L.

Suppose, for contradiction, that there is a homeomorphism $\varphi : U_L \to \mathbb{R}^2$ that is L-Lipschitz and whose inverse is also L-Lipschitz, with $J(\varphi) = f_L$ on U_L. Observe that this implies

$$\frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|} \geq \frac{1}{L}.$$

Due to our first-step construction, there exits j such that $I_j^1 = [x_{1,j}, y_{1,j}]$ for which

$$\frac{\|\varphi(x_{1,j}) - \varphi(y_{1,j})\|}{\|x_{1,j} - y_{1,j}\|} \geq \frac{\mu}{L}.$$

Repeating this logic in the second step yields another segment $I_{2,j}$ where the Lipschitz distortion is at least μ^2/L. After k such iterations, we obtain points u, v with

$$\frac{\|\varphi(u) - \varphi(v)\|}{\|u - v\|} \geq \frac{\mu^k}{L} > L,$$

which violates the assumption that φ is L-Lipschitz. Thus such a bi-Lipschitz map φ cannot exist.

Since each U_L admits a function f_L obstructing any L-Lipschitz homeomorphism of that specific type, and since these sets U_L cover the construction needed for an (a, b)-net outside the standard lattice structure, we conclude there is indeed an (a, b)-net in $\mathbb{R}^2$ not bi-Lipschitz equivalent to $\mathbb{Z}^2$ by the Theorem 8.

□

5 The case of infinite-dimensional Banach spaces

Definition 12 (Density character)**.** *The density character of a metric space (X, d) is the least cardinality of a dense subset of X.*

Theorem 10. *Any net in an infinite-dimensional Banach space X has cardinality equal to the density character of X. In particular, any two nets have the same cardinality.*

Proof. Let $\mathcal{N}$ be a net in an infinite-dimensional Banach space X and $dc(X)$ the density character of X. In this proof, have 2 directions:

- We prove $|\mathcal{N}| \leq dc(X)$:

  By the definition of density character, there exists a subset C in X such that $\overline{C} = X$ and $|C| = dc(X)$. We construct a family of disjoint sets, $\{B(n, \frac{a}{2})\}_{n\in\mathcal{N}}$. Because C is dense in X, we have the intersection of any nonempty open set and C isn't equal to the empty set, (i.e, let $\mathcal{O}$ be a nonempty open set in X, we have $\mathcal{O} \cap C \neq \phi$.)

  Now we choose $d_u \in B(u, \frac{a}{2}) \cap C$ and define a map:

$$\begin{aligned} \psi : \mathcal{N} &\to C \\ u &\mapsto d_u \end{aligned}$$

  which is well-defined. Then ψ is an injection since $\mathcal{N}$ is a-separated, so we have

$$|\mathcal{N}| = |Im(\psi)| \leq |C| = dc(X)$$

- We prove $|\mathcal{N}| \geq dc(X)$:

  We begin with 2 claims first.

  Claim 2. *All finite rational linear combinations of $\mathcal{N}$ are dense in X. i.e,*

$$\overline{Span_{\mathbb{Q}}(\mathcal{N})} = \overline{\{\textit{all finite rational linear combinations of } \mathcal{N}\}} = X$$

  with rational linear combination $= \sum_{i=1}^{m} \alpha_i n_i$ for some $m \in \mathbb{N}$, $\alpha_i \in \mathbb{Q}, n_i \in \mathcal{N}$.

  Proof. If $\overline{Span_{\mathbb{Q}}(\mathcal{N})} \neq X$, let $v \in X \setminus \overline{Span_{\mathbb{Q}}(\mathcal{N})}$. We have, for all $t \in \mathbb{Q}$:

$$t \cdot d(v, \overline{Span_{\mathbb{Q}}(\mathcal{N})}) = d(tv, \overline{Span_{\mathbb{Q}}(\mathcal{N})}) \leq d(tv, \mathcal{N}) \leq b$$

  We have $d(v, \overline{Span_{\mathbb{Q}}(\mathcal{N})}) \leq \frac{b}{t}$, take $t \to +\infty$, $d(v, \overline{Span_{\mathbb{Q}}(\mathcal{N})}) = 0$. Contradiction. □

Claim 3.

$$Card(Span_{\mathbb{Q}}(\mathcal{N})) = |\mathcal{N}|$$

Proof. This is because to every element of $Span_{\mathbb{Q}}(\mathcal{N})$, we can associate a finite subset of $\mathcal{N}$ and a finite subset of the rationals. It is then standard to construct an injection into $\mathcal{N}$. The converse inequality is trivial. □

Thanks to the two claims, we have:

$$Card(Span_{\mathbb{Q}}(\mathcal{N})) = |\mathcal{N}| \geq |C| = dc(X)$$

because C is the smallest dense subset of X. The proof is now complete by the Schroeder–Bernstein theorem. □

Lemma 7. *Let X be a metric space, and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two nets in X. Let $T : \mathcal{N}_1 \to \mathcal{N}_2$ be a bijective map by the Theorem 10, such that $\sup\{d(x, Tx) : x \in \mathcal{N}_1\} < \infty$. Then T is a bi-Lipschitz equivalence between the two nets.*

Proof. It's enough to prove that T^{-1} is a Lipschitz map because of the symmetry.

Let $\mathcal{N}_2$ be an (a, b)-net in X, and put $c = \sup\{d(x, Tx) : x \in \mathcal{N}_1\}$. Fix two points $x \neq y$ in $\mathcal{N}_1$, so we have 2 cases to discuss :

- If $d(x, y) \leq 4c$, thanks to the a-separated property of $\mathcal{N}_2$ and $\frac{d(x,y)}{4c} \leq 1$, then

$$d(Tx, Ty) \geq a \geq a\frac{d(x, y)}{4c} = \frac{a}{4c}d(x, y)$$

- If $d(x, y) \geq 4c \iff \frac{d(x,y)}{2} \geq 2c$, then

$$d(Tx, Ty) \geq d(x, y) - [d(x, Tx) + d(y, Ty)] \geq d(x, y) - 2c \geq \frac{d(x, y)}{2}$$

Hence T^{-1} is Lipschitz with constant $\max\{2, \frac{4c}{a}\}$ □

Theorem 11 (Lindenstrauss, Matoušková, Preiss)**.** *Let E be an infinite-dimensional Banach space. Then any two nets in E are Bi-lipschitz equivalent.*

Proof. Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are both (a, b)-nets in E. Our goal is to construct a bijection $T : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\sup_{x \in \mathcal{N}_1} \|x - Tx\| < \infty$. By previous lemma, this will imply that T is a bi-Lipschitz equivalence.

Let $\{y_\alpha\}_{\alpha \in I}$ be a maximal $5b$-separated subset of E. This means:

- **($5b$-separated)** $\|y_\alpha - y_\beta\| \geq 5b$ for all $\alpha \neq \beta$.

- **(Maximal)** The set $\{y_\alpha\}$ is maximal with respect to this property (i.e., adding any new point would violate the separation).

Using this family $\{y_\alpha\}_{\alpha\in I}$, we decompose E into disjoint subsets $\{C_\alpha\}$ as follows:

$$\|x - y_\alpha\| \;\le\; 5b \quad\Longrightarrow\quad x \in C_\alpha,$$

and assign x to exactly one C_α. We do it in such a way that each C_α remains disjoint from all C_β ($\beta \neq \alpha$), and each C_α has $\operatorname{diam}(C_\alpha) \le 10b$.

This is feasible precisely because $\{y_\alpha\}$ is maximal $5b$-separated. The maximality implies that for each x, there exists such an α and moreover, for any $x' \in C_\alpha$ we have

$$\|x - x'\| \;\le\; \|x - y_\alpha\| + \|y_\alpha - x'\| \;\le\; 5b + 5b \;=\; 10b.$$

Thus each C_α is of diameter at most $10b$, and the sets $\{C_\alpha\}$ form a partition of E.

Now observe that for each index α, the intersections

$$C_\alpha \cap \mathcal{N}_1 \quad\text{and}\quad C_\alpha \cap \mathcal{N}_2$$

are both infinite sets. Indeed, $\mathcal{N}_1$ and $\mathcal{N}_2$ are (a,b)-nets in an infinite-dimensional Banach space, so in "big enough" subsets of E (such as these C_α of diameter at least $5b$), each net has infinitely many points. Crucially, by arguments in the spirit of Theorem 10 on density, these sets $\big(C_\alpha \cap \mathcal{N}_i\big)$ have the same cardinality as the density character of E. Hence

$$\big|\,C_\alpha \cap \mathcal{N}_1\,\big| \;=\; \big|\,C_\alpha \cap \mathcal{N}_2\,\big|$$

so we can define a bijection

$$T_\alpha : \quad C_\alpha \cap \mathcal{N}_1 \;\longrightarrow\; C_\alpha \cap \mathcal{N}_2.$$

Do this for each α.

We now assemble these partial bijections T_α into one global mapping $T : \mathcal{N}_1 \to \mathcal{N}_2$ by the rule

$$T(x) \;=\; T_\alpha(x) \quad\text{for}\quad x \in C_\alpha \cap \mathcal{N}_1.$$

This is well-defined since each point $x \in \mathcal{N}_1$ lies in exactly one C_α.

Next, for each $x \in \mathcal{N}_1$, because x and Tx both lie in the same block C_α, we deduce

$$\|x - Tx\| \;\le\; \operatorname{diam}(C_\alpha) \;\le\; 10b.$$

Hence

$$\sup_{x\in\mathcal{N}_1} \|x - Tx\| \;\le\; 10b \;<\; \infty.$$

The proof is now complete by Lemma 7. □

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